Strand 4: Mathematically modelling a changing world

Solving Recurrence Relations and Difference Equations

Linear Mapping:

If V and W are vector spaces then a mapping $f : V \to W$ is a **Linear Mapping** if for $\boldsymbol{u}_1, \boldsymbol{u}_2 \in V$ and scalars c, c_1 and c_2

$$f(\boldsymbol{u}_1 + \boldsymbol{u}_2) = f(\boldsymbol{u}_1) + f(\boldsymbol{u}_2)$$

$f(c\,\boldsymbol{u}_1) = c\,f(\boldsymbol{u}_1)$ This condition implies homogeneity of degree 1.

or

$$f(c_1\,\boldsymbol{u}_1 + c_2\,\boldsymbol{u}_2) = c_1\,f(\boldsymbol{u}_1) + c_2\,f(\boldsymbol{u}_2)$$

Note if $f(x) = x^2$

$\Rightarrow f(cx) = (cx)^2 = c^2 x^2 = c^2 f(x)$ This is not a Linear Mapping

The Linear Difference Operator Δ

$$\Delta u_n = u_{n+1} - u_n$$

$$\Delta u_n + u_n = u_{n+1}$$

$$\Rightarrow \quad \boxed{(\Delta + 1)u_n = u_{n+1}}$$

$$(\Delta+1)\{(\Delta+1)u_n\} = (\Delta+1)(u_{n+1}) = u_{n+2}$$

$$\Rightarrow \quad (\Delta+1)^2 u_n = u_{n+2}$$

$$\Rightarrow \quad (\Delta+1)^3 u_n = u_{n+3}$$

$$\Rightarrow \quad \boxed{(\Delta+1)^r u_n = u_{n+r}}$$

Consider the **Recurrence Relation**

$$u_{n+1} = \alpha\, u_n \qquad \Rightarrow \qquad u_n = A\alpha^n$$

Or the **Difference Equation**

$$\boxed{\begin{array}{c}(\Delta+1)u_n = \alpha\, u_n \\ \text{or} \\ \{(\Delta+1) - \alpha\}u_n = 0\end{array}} \qquad \Rightarrow \qquad \boxed{u_n = A\alpha^n}$$

If $u_{n+2} + l\,u_{n+1} + m\,u_n = f(n)$ where l and m are constants

The following are Particular Solutions to the Non Homogeneous equation

$f(n)$	Particular Solution
Constant	Constant
n	$a + bn$
n^2	$a + bn + cn^2$
k^n	ak^n (or ank^n in special cases)

where a, b and c are constants

Solving Second Order Homogeneous Equations

1. When the Characteristic Equation has **distinct** roots.

$$u_{n+2} + a\,u_{n+1} + b\,u_n = 0$$ and its charactistic polynomial

$$x^2 + a\,x + b\,x = 0$$ with roots α, β where $\alpha \neq \beta$

$\Rightarrow$ $$\left\{(\Delta+1)^2 + a(\Delta+1) + b\right\}u_n = 0$$ a quadratic in $(\Delta+1)$ and a second order

Difference Equation which can be factorised.

$$\left\{(\Delta+1)-\alpha\right\}\left\{(\Delta+1)-\beta\right\}u_n = 0 = 0$$

$$\left\{(\Delta+1)-\alpha\right\}v_n = 0 \quad \text{where} \quad v_n = \left\{(\Delta+1)-\beta\right\}u_n$$

$$v_n = C\alpha^n$$

$$v_n = \left\{(\Delta+1)-\beta\right\}u_n = C\alpha^n$$

This is a Non Homogeneous equation with a particular solution $D\alpha^n$ or $A\alpha^{n+1}$ and a solution $E\beta^n$ or $B\beta^{n+1}$ to the Homogenous equation.

$$u_{n+1} - \beta\,u_n = C\alpha^n$$

$$u_{n+1} = \beta\,u_n + C\alpha^n$$

$$u_{n+1} = \beta\left(\beta\,u_{n-1} + C\alpha^{n-1}\right) + C\alpha^n$$

$$u_{n+1} = \beta^2 u_{n-1} + \beta\left(C\alpha^{n-1}\right) + C\alpha^n$$

$$u_{n+1} = \beta^2\left(\beta\,u_{n-2} + C\alpha^{n-2}\right) + \beta\left(C\alpha^{n-1}\right) + C\alpha^n$$

$$u_{n+1} = \beta^2\left(\beta\,u_{n-2} + C\alpha^{n-2}\right) + \beta\left(C\alpha^{n-1}\right) + C\alpha^n$$

$$u_{n+1} = \beta^3 u_{n-2} + \beta^2\left(C\alpha^{n-2}\right) + \beta\left(C\alpha^{n-1}\right) + C\alpha^n$$

$$u_{n+1} = \beta^n u_1 + \beta^{n-1}\left(C\alpha^1\right) + \beta^{n-2}\left(C\alpha^2\right) + \ldots\ldots\ldots\ldots + \beta\left(C\alpha^{n-1}\right) + C\alpha^n$$

$$u_{n+1} = \beta^n u_1 + C\beta^n\left\{\left(\frac{\alpha}{\beta}\right)^1 + \left(\frac{\alpha}{\beta}\right)^2 + \ldots\ldots\ldots\ldots + \left(\frac{\alpha}{\beta}\right)^n\right\}$$

$$u_{n+1} = \beta^n u_1 + C\beta^n\left\{\left(\frac{\alpha}{\beta}\right)\frac{\left(\frac{\alpha}{\beta}\right)^n - 1}{\left(\frac{\alpha}{\beta}\right) - 1}\right\}$$

Note $\left(\frac{\alpha}{\beta}\right) - 1 \neq 0$ as $\alpha \neq \beta$

$$\boxed{u_{n+1} = A\alpha^{n+1} + B\beta^{n+1}}$$ for some constants A, B

Solving Second Order Homogeneous Equations

2. When the Characteristic Equation has **identical** roots

$$u_{n+2} + a\,u_{n+1} + b\,u_n = 0$$ and its charactistic polynomial

$$x^2 + a\,x + b\,x = 0$$ with two identical roots α

$$\{(\Delta+1)-\alpha\}\{(\Delta+1)-\alpha\}u_n = 0 = 0$$

$$\{(\Delta+1)-\alpha\}v_n = 0 \quad \text{where} \quad v_n = \{(\Delta+1)-\alpha\}u_n$$

$$v_n = C\alpha^n$$

$$v_n = \{(\Delta+1)-\alpha\}u_n = C\alpha^n$$

This is a Non Homogeneous equation with a particular solution $C\,n\alpha^n$ or $B\,n\alpha^{n+1}$ and a solution $D\,\alpha^n$ or $A\,\alpha^{n+1}$ to the Homogenous equation.

$$u_{n+1} - \alpha\,u_n = C\alpha^n$$

$$u_{n+1} = \alpha\,u_n + C\alpha^n$$

$$u_{n+1} = \alpha\left(\alpha\,u_{n-1} + C\alpha^{n-1}\right) + C\alpha^n$$

$$u_{n+1} = \alpha^2 u_{n-1} + C\alpha^n + C\alpha^n = \alpha^2 u_{n-1} + 2C\alpha^n$$

$$u_{n+1} = \alpha^2\left(\alpha\,u_{n-2} + C\alpha^{n-2}\right) + 2C\alpha^n$$

$$u_{n+1} = \alpha^3 u_{n-2} + 3C\alpha^n$$

$$u_{n+1} = \alpha^n u_1 + nC\alpha^n = D\alpha^n + nC\alpha^n$$

$$\boxed{u_{n+1} = A\alpha^{n+1} + nB\alpha^{n+1} = (A+nB)\alpha^{n+1}}$$